

Complex interpolation and twisted Hilbert spaces

Félix Cabello Sánchez, Jesús M. F. Castillo, and Nigel J. Kalton [†]

ABSTRACT. We show that Rochberg’s generalized interpolation spaces $\mathcal{X}^{(n)}$ arising from analytic families of Banach spaces form exact sequences $0 \rightarrow \mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(n+k)} \rightarrow \mathcal{X}^{(k)} \rightarrow 0$. We study some structural properties of those sequences; in particular, we show that nontriviality, having strictly singular quotient map, or having strictly cosingular embedding depend only on the basic case $n = k = 1$. If we focus on the case of Hilbert spaces obtained from the interpolation scale of ℓ_p spaces, then $\mathcal{X}^{(2)}$ becomes the well-known Kalton-Peck Z_2 space; we then show that $\mathcal{X}^{(n)}$ is (or embeds in, or is a quotient of) a twisted Hilbert space only if $n = 1, 2$, which solves a problem posed by David Yost; and that it does not contain ℓ_2 complemented unless $n = 1$. We construct another nontrivial twisted sum of Z_2 with itself that contains ℓ_2 complemented.

1. Introduction

In 1979, Kalton and Peck developed a method to produce nontrivial self-extensions of most quasi-Banach spaces with unconditional basis [14, Section 4], including all Banach spaces apart from c_0 ; see [3, Theorem 1]. The most glaring examples are perhaps the so-called Z_p spaces, which are twisted sums of the ℓ_p spaces. If, however, one wants to construct twisted sums of Z_p , the Kalton-Peck’s method simply does not work because of their poor unconditional structure. On the other hand, the existence of such twisted sums is guaranteed by the local theory of exact sequences, at least when $p > 1$; see e.g., [2]. Our starting goal with this paper was to develop a method to obtain twisted sums of twisted sum spaces, keeping the Z_p spaces as the control case.

The path connecting interpolation theory and twisted sums was opened by Rochberg and Weiss, who introduce in [19] certain spaces which naturally arise in the study of “analytic families” of Banach spaces and that turn out to be twisted sums of the “intermediate” spaces. Actually, if \mathcal{F} is the usual Calderón space of analytic functions on the strip $0 < \Re z < 1$ associated to the couple (ℓ_∞, ℓ_1) in the complex interpolation method then, as it is well-known, $[\ell_\infty, \ell_1]_\theta = \{f(\theta) : f \in \mathcal{F}\} = \ell_p$, where $p = 1/\theta$ and $0 < \theta < 1$ and

$$(1) \quad Z_p = \{(f'(\theta), f(\theta)) : f \in \mathcal{F}\},$$

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with the quotient norm inherited from \mathcal{F} —though this is not made explicit in [19].

Nothing seems to prevent one from adding more derivatives to (1) and figure out that the resulting space represents the iterated twisted sum spaces. Such is exactly what Rochberg did in [20], in the broader setting of analytic families of Banach spaces. Performing that is not, by far, as simple as it sounds; and perhaps the turning point in Rochberg’s approach is the using of Taylor coefficients instead of merely putting derivatives, as it is suggested in [13, Section 10, p. 1161].

Such approach is the one we adopt in this paper, which in this regard can well be considered a spin-off from Rochberg’s [20]; with several variations, the first of which is the use of admissible spaces of analytic functions instead of analytic families, what makes “reiteration” both unavailable and unnecessary. Thus, given an admissible space of analytic functions \mathcal{F} , we consider the space $\mathcal{X}^{(n)}$ of all possible lists of Taylor coefficients of functions in \mathcal{F} of length n – at a fixed point z which is understood from now on – endowed with the obvious infimum norm on it. Then we observe that if $m = n + k$, there is an exact sequence

$$(2) \quad 0 \longrightarrow \mathcal{X}^{(n)} \longrightarrow \mathcal{X}^{(m)} \longrightarrow \mathcal{X}^{(k)} \longrightarrow 0$$

and so $\mathcal{X}^{(m)}$ is a twisted sum of $\mathcal{X}^{(n)}$ and $\mathcal{X}^{(k)}$. The key nontrivial step here is obtaining the right form of the embedding. To this we devote section 3 in which we obtain two (equivalent) representations for the embedding, depending on the representation of the spaces. Regarding the sequences themselves, we will show that many properties, such as nontriviality, having strictly singular quotient map, or having strictly cosingular embedding depend only on the seed case $n = k = 1$. The nontriviality of this case has to be worked apart.

We then focus on the case in which \mathcal{F} is the Calderón space of the couple (ℓ_∞, ℓ_1) . If we fix $z = \frac{1}{2}$, then $\mathcal{X}^{(1)} = \ell_2$ and $\mathcal{X}^{(2)}$ is the Kalton-Peck Z_2 space [14]. The space $\mathcal{X}^{(3)}$ is both a twisted sum of ℓ_2 with Z_2 and a twisted sum of Z_2 with ℓ_2 , and $\mathcal{X}^{(4)}$ is, among other possibilities, a twisted sum of Z_2 with itself, as desired. We then pass to establish structural properties of the spaces $\mathcal{X}^{(n)}$ and of the sequences (2). Regarding the spaces, we will show that $\mathcal{X}^{(n)}$ is (or embeds in, or is a quotient of) a twisted Hilbert space only if $n = 1, 2$ –which solves a problem posed by David Yost– and that it does not contain ℓ_2 complemented unless $n = 1$. To put this result in perspective, we will construct a nontrivial twisted sum of Z_2 with itself that contains ℓ_2 complemented.

2. Preliminaires

We warmly recommend the reader who is not familiar with Kalton and Peck paper [14] or Rochberg’s [20] to postpone this article until get acquainted with them. Perusing the papers [9, 21], the article [13] in the Handbook and the monograph [5] can help with the background. Anyway, the basic ingredients to read this paper are operatively described next.

2.1. Exact sequences. A short sequence of Banach spaces and (linear, bounded) operators

$$(3) \quad 0 \longrightarrow A \xrightarrow{I} B \xrightarrow{Q} C \longrightarrow 0$$

is said to be exact if the kernel of each arrow equals the image of the preceding one. As $I(A)$ is closed in B the operator I embeds A as a subspace of B and C is isomorphic to the quotient $B/I(A)$, by the open mapping theorem. For this reason one often says that B is a twisted sum of A with C (in that order); the whole sequence (3) is said to be an extension of C by A (the order was reversed by “functorial” reasons).

The extension (3) is said to be trivial if there is an operator $P : B \rightarrow A$ such that $P \circ I = \mathbf{1}_A$ (i.e., $I(A)$ is complemented in B); equivalently, there is an operator $J : C \rightarrow B$ such that $Q \circ J = \mathbf{1}_C$. In this case $P \times Q : B \rightarrow A \times C$ is an isomorphism, with inverse $I \oplus J$ and thus the “twisted sum” B is (isomorphic to) the direct sum $A \oplus C = A \times C$.

2.2. Admissible spaces of analytic functions. We will work within the framework of an admissible space of analytic functions as defined by Kalton and Montgomery-Smith in [13, Section 10]. So, let U be an open set of \mathbb{C} conformally equivalent to the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and W a complex Banach space. A Banach space \mathcal{F} of analytic functions $F : U \rightarrow W$ is said to be admissible provided:

- (a) For each $z \in U$, the evaluation map $\delta_z : \mathcal{F} \rightarrow W$ is bounded.
- (b) If $\varphi : U \rightarrow \mathbb{D}$ is a conformal equivalence, then $F \in \mathcal{F}$ if and only if $\varphi \cdot F \in \mathcal{F}$ and $\|\varphi \cdot F\|_{\mathcal{F}} = \|F\|_{\mathcal{F}}$.

For each $z \in U$ we define $X_z = \{x \in W : x = F(z) \text{ for some } F \in \mathcal{F}\}$ with the norm $\|x\| = \inf\{\|F\|_{\mathcal{F}} : x = F(z)\}$ so that X_z is isometric to $\mathcal{F}/\ker \delta_z$ with the quotient norm. One often says that $(X_z)_{z \in U}$ is an analytic family of Banach spaces. The simplest examples arise from complex interpolation. Indeed, let (X_0, X_1) be a Banach couple and take $W = X_0 + X_1$ and U the strip $0 < \Re z < 1$. Let $\mathcal{F} = \mathcal{C}(X_0, X_1)$ be the Calderón space of those continuous functions $F : \overline{U} \rightarrow W$ which are analytic on U and satisfy the boundary conditions that, for $k = 0, 1$ one has $F(k + ti) \in X_k$ and $\|F\|_{\mathcal{C}} = \sup\{\|F(k + ti)\|_{X_k} : t \in \mathbb{R}, k = 0, 1\} < \infty$. Then \mathcal{F} is admissible and $X_z = [X_0, X_1]_{\theta}$, with $\theta = \Re z$, is an analytic family.

It is important now to realize that when \mathcal{F} is admissible, then the map $\delta_z^n : \mathcal{F} \rightarrow W$, evaluation of the n -th derivative at z , is bounded for all $z \in U$ and all $n \in \mathbb{N}$ by an iterated use of (a), the definition of derivative and the principle of uniform boundedness. Thus, it makes sense to consider the Banach spaces

$$(4) \quad \mathcal{F} / \bigcap_{i < n} \ker \delta_z^i \quad (n \in \mathbb{N}).$$

3. Exact sequences of derived spaces

3.1. Lists of Taylor coefficients. Following Rochberg, let us fix $z \in U$ and consider the following spaces:

$$\mathcal{X}_z^{(n)} = \{(x_{n-1}, \dots, x_0) \in W^n : x_i = \widehat{f}[i; z] \text{ for some } f \in \mathcal{F} \text{ and all } 0 \leq i < n\},$$

where $\widehat{f}[i; z] = f^{(i)}(z)/i!$ is the i -th Taylor coefficient of f at z . Thus, the elements of $\mathcal{X}_z^{(n)}$ are (truncated) sequences of Taylor’s coefficients (at z) of functions in \mathcal{F} arranged in decreasing order. Here, we deviate from Rochberg notation in two points: first, the superscript (n) refers to

the “number of variables” and not to the highest derivative, and second, we have arranged Taylor coefficients decreasingly in order to match with the usual notation for twisted sums, with the subspace on the left and the quotient on the right. If we equip $\mathcal{X}_z^{(n)}$ with the obvious quotient norm, it is isometric to $\mathcal{F} / \bigcap_{i < n} \ker \delta_z^i$ via Taylor coefficients and so it is complete. From now on we shall omit the base point z , which is understood to be fixed.

3.2. Operators. We introduce next certain “natural” operators linking the various spaces $\mathcal{X}^{(n)}$ as n varies. Those operators will be used to construct the exact sequences we want.

To this end, for $1 \leq n, k < m$ we denote by $\iota_{n,m} : W^n \rightarrow W^m$ the inclusion on the left given by $\iota_{n,m}(x_n, \dots, x_1) = (x_n, \dots, x_1, 0, \dots, 0)$ and by $\pi_{m,k} : W^m \rightarrow W^k$ the projection on the right given by $\pi_{m,k}(x_m, \dots, x_k, \dots, x_1) = (x_k, \dots, x_1)$. While $\pi_{m,k}$ is obviously a quotient map from $\mathcal{X}^{(m)}$ onto $\mathcal{X}^{(k)}$, it is not clear at all that $\iota_{n,m}$ maps $\mathcal{X}^{(k)}$ to $\mathcal{X}^{(n)}$, let alone its continuity. To prove that this is indeed the case we need some extra work.

Observe that if φ is as in (b) and if ϕ is a “polynomial” in φ , that is, $\phi = \sum_i a_i \varphi^i$ for some finite sequence of complex numbers (a_i) , then $\phi \cdot f \in \mathcal{F}$ for each $f \in \mathcal{F}$ and $\|\phi \cdot f\|_{\mathcal{F}} \leq (\sum_i |a_i|) \|f\|_{\mathcal{F}}$.

LEMMA 1. *Let $\varphi : U \rightarrow \mathbb{D}$ be a conformal equivalence vanishing at z . Then, for $0 \leq k \leq m$ there is a polynomial P of degree at most m such that $\widehat{P \circ \varphi}[i; z] = \delta_{ik}$ for every $0 \leq i \leq m$.*

PROOF. If $f : U \rightarrow \mathbb{C}$ is holomorphic, then $f \circ \varphi^{-1}$ is holomorphic on the disk and we have

$$f(\varphi^{-1}(w)) = \sum_{n=0}^{\infty} a_n w^n \quad (|w| < 1),$$

where a_n is the n -th Taylor coefficient of $f \circ \varphi^{-1}$ at the origin. In particular $f \circ \varphi^{-1}$ has a contact of order m with the polynomial defined by $P(w) = \sum_{n=0}^m a_n w^n$ at the origin. As φ is a conformal equivalence we have that $f = f \circ \varphi^{-1} \circ \varphi$ has a contact of order m with the function

$$P \circ \varphi = \sum_{n=0}^m a_n \varphi^n$$

at $z = \varphi^{-1}(0)$. In particular the first m derivatives of f and $\sum_{n=0}^m a_n \varphi^n$ agree at z . The Lemma follows just applying this construction to the function $f(w) = (w - z)^k$. \square

The following Proposition is a slight generalization of [20, Proposition 3.1]:

PROPOSITION 1. *Suppose $1 \leq n, k < m$. Then:*

- (a) *The map $\iota_{n,m} : \mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(m)}$ is bounded.*
- (b) *The map $\pi_{m,k} : \mathcal{X}^{(m)} \rightarrow \mathcal{X}^{(k)}$ is an “isometric” quotient.*

PROOF. Part (b) is obvious. To prove (a) we must prove that there is a constant M such that if (x_n, \dots, x_1) is the list of Taylor coefficients of some $f \in \mathcal{F}$, then there is another $g \in \mathcal{F}$ whose coefficients are $(x_n, \dots, x_1, 0, \dots, 0)$ with $\|g\|_{\mathcal{F}} \leq M \|f\|_{\mathcal{F}}$. Set $k = m - n$ and apply Lemma 1 to get a polynomial $\phi = \sum_{i=0}^{m-1} a_i \varphi^i$ such that $\widehat{\phi}[i; z] = \delta_{ik}$, for $0 \leq i < n + k$ and take

$M = \sum_{i=0}^{n+k-1} |a_i|$. Now, if $f \in \mathcal{F}$ and $g = \phi f$, then $\|g\|_{\mathcal{F}} \leq M\|f\|_{\mathcal{F}}$. Moreover, for $i \in [0, n+k)$, one has

$$\widehat{g}[i; z] = (\widehat{\phi f})[i; z] = \sum_{j=0}^i \widehat{\phi}[j; z] \cdot \widehat{f}[i-j; z],$$

by Leibniz rule. Hence $\widehat{g}[i; z] = 0$ if $i < k$ and for $i \geq k$ we have $\widehat{g}[i; z] = \widehat{f}[i-k; z]$, as required. \square

3.3. Exactness. From now on we will omit the names $\iota_{m,n}$ and $\pi_{m,k}$ and so unlabelled arrows $\mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(m)}$ must be understood to be $\iota_{n,m}$ if $n \leq m$ and $\pi_{n,m}$ for $n \geq m$, unless otherwise declared. With these conventions the aim of this Section is to prove that, given integers n and k , the “obvious” sequence $0 \rightarrow \mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(n+k)} \rightarrow \mathcal{X}^{(k)} \rightarrow 0$ is exact.

First of all, observe that the various possible sequences passing through a given $\mathcal{X}^{(m)}$ are compatible in the sense that if $m = k + n = i + j$, with $k < i$ say, then the following diagram is commutative

$$(5) \quad \begin{array}{ccccc} \mathcal{X}^{(j)} & \xlongequal{\quad} & \mathcal{X}^{(j)} & & \\ \downarrow & & \downarrow & & \\ \mathcal{X}^{(n)} & \longrightarrow & \mathcal{X}^{(m)} & \longrightarrow & \mathcal{X}^{(k)} \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{X}^{(n-j)} & \longrightarrow & \mathcal{X}^{(i)} & \longrightarrow & \mathcal{X}^{(k)} \end{array}$$

The key point is isolated in the next lemma.

LEMMA 2. *If $(x, 0, \dots, 0) \in \mathcal{X}^{(k+1)}$, then $x \in \mathcal{X}^{(1)}$.*

PROOF. Pick $x \in W$ and suppose $(x, 0, \dots, 0) \in \mathcal{X}^{(k+1)}$. Let us take $f \in \mathcal{F}$ such that $\widehat{f}[i; z] = 0$ for $i < k$ and $x = \widehat{f}[k; z]$. Then f has a zero of order $k-1$ at z and it can be written as $f = \varphi^k g$, where $g : U \rightarrow W$ is analytic. It follows from (b) that $g \in \mathcal{F}$ and $\|g\|_{\mathcal{F}} = \|f\|_{\mathcal{F}}$. But

$$x = \widehat{f}[k; z] = (\widehat{\varphi^k g})[k; z] = \sum_{i=0}^k (\widehat{\varphi^k})[i; z] \cdot \widehat{g}[k-i; z] = \frac{(\varphi^k)^{(k)}(z)g(z)}{k!} = \varphi'(z)^k g(z)$$

and so $x \in \mathcal{X}^{(1)}$. \square

THEOREM 1. *The sequence $0 \rightarrow \mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(n+k)} \rightarrow \mathcal{X}^{(k)} \rightarrow 0$ is exact.*

PROOF. The proof proceeds by induction on $m = n + k$. The previous lemma shows that for every $m \in \mathbb{N}$, the sequence $0 \rightarrow \mathcal{X}^{(1)} \rightarrow \mathcal{X}^{(m)} \rightarrow \mathcal{X}^{(m-1)} \rightarrow 0$ is exact. By the induction hypothesis, the sequence $0 \rightarrow \mathcal{X}^{(n-1)} \rightarrow \mathcal{X}^{(m-1)} \rightarrow \mathcal{X}^{(k)} \rightarrow 0$ is also exact.

The compatibility of such sequences yields the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{X}^{(1)} & \xlongequal{\quad} & \mathcal{X}^{(1)} & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{X}^{(n)} & \longrightarrow & \mathcal{X}^{(m)} & \longrightarrow & \mathcal{X}^{(k)} \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathcal{X}^{(n-1)} & \longrightarrow & \mathcal{X}^{(m-1)} & \longrightarrow & \mathcal{X}^{(k)} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and a simple chasing of arrows shows that the middle sequence must also be exact. \square

COROLLARY 1. *If $(x_n, \dots, x_1, 0, \dots, 0) \in \mathcal{X}^{(n+k)}$, then $(x_n, \dots, x_1) \in \mathcal{X}^{(n)}$.*

This implies that $\|(x_n, \dots, x_1, 0, \dots, 0)\|_{\mathcal{X}^{(n+k)}}$ is equivalent to $\|(x_n, \dots, x_1)\|_{\mathcal{X}^{(n)}}$, although we will not pursue any bound here.

A new look can be paid now at Diagram (5) to exploit its form to study the splitting of the exact sequences it contains. After Theorem 1 the diagram has become

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{X}^{(j)} & \xlongequal{\quad} & \mathcal{X}^{(j)} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{X}^{(n)} & \longrightarrow & \mathcal{X}^{(m)} & \longrightarrow & \mathcal{X}^{(k)} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathcal{X}^{(n-j)} & \longrightarrow & \mathcal{X}^{(i)} & \longrightarrow & \mathcal{X}^{(k)} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Thus, if the middle horizontal sequence splits, then so does the lower one, and if the middle vertical sequence splits, then so does the vertical sequence on the left. Putting together these

two pieces one gets the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{X}^{(n)} & \longrightarrow & \mathcal{X}^{(n+k)} & \longrightarrow & \mathcal{X}^{(k)} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & \mathcal{X}^{(1)} & \longrightarrow & \mathcal{X}^{(k+1)} & \longrightarrow & \mathcal{X}^{(k)} & \longrightarrow & 0 \\
& & \parallel & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{X}^{(1)} & \longrightarrow & \mathcal{X}^{(2)} & \longrightarrow & \mathcal{X}^{(1)} & \longrightarrow & 0
\end{array}$$

from which it immediately follows

COROLLARY 2. *If the sequence $0 \longrightarrow \mathcal{X}^{(n)} \longrightarrow \mathcal{X}^{(n+k)} \longrightarrow \mathcal{X}^{(k)} \longrightarrow 0$ is nontrivial for $n = k = 1$, then it is nontrivial for all integers n and k .*

3.4. An isometric variant. There is another form for the exact sequences $0 \rightarrow \mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(n+k)} \rightarrow \mathcal{X}^{(k)} \rightarrow 0$ which is even easier to describe in abstract terms. Consider again the quotient spaces

$$Q_z^{(n)} = \mathcal{F} / \bigcap_{i < n} \ker \delta_z^i \quad (n \in \mathbb{N}).$$

These spaces are isometric to the corresponding $\mathcal{X}_z^{(n)}$ via Taylor coefficients, but we do not need this fact at this moment. Let us fix integers n and k . It is clear that there is a natural quotient map from $Q_z^{(n+k)}$ onto $Q_z^{(k)}$ that we shall not even label. Less obvious is that the kernel of this map is isometric to $\mathcal{X}_z^{(n)}$, although this time the isometry is not “natural”. To see this let us fix a conformal equivalence $\varphi : U \rightarrow \mathbb{D}$ having a (single) zero at z . (We observe that if ϕ is another conformal equivalence with $\phi(z) = 0$, then $\phi = \lambda\varphi$, where $\lambda \in \mathbb{T}$; thus φ is unique if we insist that $\varphi'(0)$ is real and positive.) Now recall that $f \in \bigcap_{i < k} \ker \delta_z^i$ if and only if there is a (necessarily unique) $g \in \mathcal{F}$ such that $f = \varphi^k g$ and one has $\|f\|_{\mathcal{F}} = \|g\|_{\mathcal{F}}$, by (b). It is therefore clear that the map $f \in \mathcal{F} \mapsto \varphi^k f \in \mathcal{F}$ induces an isometry of $Q_z^{(n)}$ into $Q_z^{(n+k)}$ whose range is $\ker(Q_z^{(n+k)} \rightarrow Q_z^{(k)})$.

Thus the space $Q_z^{(n+k)}$ is an “isometric” twisted sum of $Q_z^{(n)}$ and $Q_z^{(k)}$. More precisely, the short sequence

$$(7) \quad 0 \longrightarrow Q_z^{(n)} \xrightarrow{\varphi^k \cdot} Q_z^{(n+k)} \longrightarrow Q_z^{(k)} \longrightarrow 0$$

is exact. We will omit from now on the base point z , which is understood. As before, the decompositions of a given $Q_z^{(m)}$ into twisted sum of the preceding spaces $Q_z^{(n)}$ are all compatible

in the sense that if $m = k + n = i + j$, with $k < i$, then the following diagram is commutative

$$\begin{array}{ccccc}
 Q^{(j)} & \xlongequal{\quad} & Q^{(j)} & & \\
 \varphi^{j-n} \downarrow & & \downarrow \varphi^i & & \\
 Q^{(n)} & \xrightarrow{\varphi^k} & Q^{(m)} & \longrightarrow & Q^{(k)} \\
 \downarrow & & \downarrow & & \parallel \\
 Q^{(j-n)} & \xrightarrow{\varphi^k} & Q^{(i)} & \longrightarrow & Q^{(k)}
 \end{array}$$

It is interesting to compare the sequence (7) to that appearing in Theorem 1. To this end, we observe that, after identifying $\mathcal{X}^{(m)}$ and $Q^{(m)}$ through Taylor coefficients, the operator $\mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(n+k)}$ which corresponds to $\iota_{n,n+k} : \mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(n+k)}$ is just multiplication by ϕ , where ϕ is the polynomial appearing in the proof of Proposition 1(a), that is, $\phi = \sum_{0 \leq i < n+k} a_i \varphi^i$, with $\widehat{\phi}[i; z] = \delta_{ik}$ for $0 \leq i < n+k$. Clearly, $a_i = 0$ for $0 \leq i < k$ and so $\phi = \varphi^k \psi$, where $\psi = \sum_{k \leq i < n+k} a_i \varphi^{i-k}$. Thus the following diagram is commutative

$$\begin{array}{ccccc}
 Q^{(n)} & \xrightarrow{\varphi^k} & Q^{(n+k)} & \longrightarrow & Q^{(k)} \\
 \parallel & & \downarrow \psi & & \parallel \\
 Q^{(n)} & \xrightarrow{\phi} & Q^{(n+k)} & \longrightarrow & Q^{(k)} \\
 \downarrow \widehat{\cdot} & & \downarrow \widehat{\cdot} & & \downarrow \widehat{\cdot} \\
 \mathcal{X}^{(n)} & \xrightarrow{\iota_{n,n+k}} & \mathcal{X}^{(n+k)} & \longrightarrow & \mathcal{X}^{(k)}
 \end{array}$$

It follows from the 3-lemma (see for instance [10, Lemma 1.1]) and the open mapping theorem that multiplication by ψ induces an automorphism of $Q^{(n+k)}$ and so, in the preceding diagram, the first row is equivalent to the second one, and both are “isomorphically equivalent” (in the language of [6, p. 256]) to the third one; which means that the three sequences have the same “isomorphic” properties.

3.5. The space $\mathcal{X}^{(n+k)}$ as a twisted sum of $\mathcal{X}^{(n)}$ and $\mathcal{X}^{(k)}$. It is a part of the by now classical theory of twisted sums as developed by Kalton (see [11, Proposition 3.3] or [14, Theorem 2.4]) that if A and C are Banach or quasi-Banach spaces, then every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ arises, up to equivalence, from a quasilinear map from C to A . Thus, in view of Theorem 1, given integers k and n , there must be some quasilinear map $\Omega_{k,n}$ associated to the exact sequence $0 \rightarrow \mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(n+k)} \rightarrow \mathcal{X}^{(k)} \rightarrow 0$. From an abstract point of view, the description of $\Omega_{k,n}$ is rather easy. One fixes some (small) $\varepsilon > 0$. Given $x = (x_{k-1}, \dots, x_0) \in \mathcal{X}^{(k)}$ we select (homogeneously) $f \in \mathcal{F}$ such that $\|f\| \leq (1 + \varepsilon)\|x\|_{\mathcal{X}^{(n)}}$ and $\widehat{f}[i; z] = x_i$ for $0 \leq i < k$ and we define $\Omega_{k,n} : \mathcal{X}^{(k)} \rightarrow W^n$ by letting

$$(8) \quad \Omega_{k,n}(x) = (\widehat{f}[n+k-1; z], \dots, \widehat{f}[k; z]).$$

Following the uses of the theory, the twisted sum space (sometimes known as the derived space) is then defined by

$$\mathcal{X}^{(n)} \oplus_{\Omega_{k,n}} \mathcal{X}^{(k)} = \{(y, x) \in W^n \times W^k : x \in \mathcal{X}^{(k)}, y - \Omega_{k,n}(x) \in \mathcal{X}^{(n)}\},$$

endowed with the quasinorm

$$(9) \quad \|(y, x)\|_{\Omega_{k,n}} = \|y - \Omega_{k,n}(x)\|_{\mathcal{X}^{(n)}} + \|x\|_{\mathcal{X}^{(k)}}.$$

Of course it has not yet been proved neither that $\Omega_{k,n}$ is quasilinear nor that the formula (9) defines a quasinorm. We may skip these steps since we have the following.

PROPOSITION 2. *The spaces $\mathcal{X}^{(n)} \oplus_{\Omega_{k,n}} \mathcal{X}^{(k)}$ and $\mathcal{X}^{(n+k)}$ are the same.*

PROOF. Suppose $(y, x) = (y_{n-1}, \dots, y_0, x_{k-1}, \dots, x_0) \in \mathcal{X}^{(n+k)}$ so that there is $F \in \mathcal{F}$ whose list of Taylor coefficients begins with (y, x) . Then $x \in \mathcal{X}^{(k)}$ and $(\Omega_{k,n}(x), x) \in \mathcal{X}^{(n+k)}$, so $(y, x) - (\Omega_{k,n}(x), x) = (y - x, \Omega_{k,n}(x), 0)$ belongs to $\mathcal{X}^{(n+k)}$ and by Lemma 2 we have $y - \Omega_{k,n}(x) \in \mathcal{X}^{(n)}$. Regarding the involved norms, one has

$$\|y - \Omega_{k,n}(x)\|_{\mathcal{X}^{(n)}} \leq C(\|(y, x)\|_{\mathcal{X}^{(n+k)}} - \|(\Omega_{k,n}(x), x)\|_{\mathcal{X}^{(n+k)}}) \leq (C + 1)\|(y, x)\|_{\mathcal{X}^{(n+k)}},$$

where C is the constant implicit in Lemma 2. Hence $\|(y, x)\|_{\Omega_{k,n}} \leq (C + 2)\|(y, x)\|_{\mathcal{X}^{(n+k)}}$.

As for the other containment, suppose $(y, x) \in \mathcal{X}^{(n)} \oplus_{\Omega_{k,n}} \mathcal{X}^{(k)}$, that is, $x \in \mathcal{X}^{(k)}$ and $y - \Omega_{k,n}(x) \in \mathcal{X}^{(n)}$. Then if f is the function associated to $\Omega_{k,n}(x)$ as in (8) and $g \in \mathcal{F}$ is almost optimal for $y - \Omega_{k,n}(x) \in \mathcal{X}^{(n)}$, taking ϕ as in Lemma 1, we have that (y, x) is the list of Taylor coefficients of $F = f + \phi \cdot g$, so $(y, x) \in \mathcal{X}^{(n+k)}$ and

$$\|(y, x)\|_{\mathcal{X}^{(n+k)}} \leq \|f + \phi \cdot g\|_{\mathcal{F}} \leq (1 + \varepsilon)(M\|y - \Omega_{k,n}(x)\|_{\mathcal{X}^{(n)}} + \|x\|_{\mathcal{X}^{(k)}}),$$

where M is as in the proof of Proposition 1(a). □

4. Singularity of the exact sequences of derived spaces

Recall that an operator is said to be strictly singular if its restriction to an infinite dimensional subspace of its domain is never an isomorphism; and that an operator $u : A \rightarrow B$ is strictly cosingular if for every infinite codimensional subspace C of B the composition $\pi \circ u : A \rightarrow B \rightarrow B/C$ fails to be onto; equivalently, $u^* : B^* \rightarrow A^*$ is not an isomorphism when restricted to any weakly* closed infinite-dimensional subspace of B^* . Strictly singular operators were introduced by Kato [15] and strictly cosingular by Pełczyński [17].

An exact sequence is said to be singular when the quotient map is strictly singular and will be called cosingular when the embedding is strictly cosingular. We refer the reader to [7, 3] for some steps into the theory of singular and cosingular sequences. The Kalton-Peck sequences $0 \rightarrow \ell_p \rightarrow Z_p \rightarrow \ell_p \rightarrow 0$ are singular for all $p \in (0, \infty)$ and cosingular at least for $p \in (1, \infty)$. We need the following result.

LEMMA 3. *Assume one has a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{I} & B & \xrightarrow{Q} & C & \longrightarrow & 0 \\ & & \downarrow t & & \downarrow T & & \parallel & & \\ 0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with exact rows. If both Q and t are strictly singular then T is strictly singular.

PROOF. We need the following characterization of strictly singular quotient maps. Let B be a Banach space and A a closed subspace of B . Then the quotient map $Q : B \rightarrow B/A$ is strictly singular if and only if for every infinite-dimensional subspace $B' \subset B$ there is an infinite-dimensional $A' \subset A$ and a compact (actually nuclear) operator $K : A' \rightarrow B$ such that $I + K$ embeds isomorphically A' into B' . This maybe folklore; see [8, Proposition 3.2] for an explicit proof. A certainly classical result establishes that an operator $t : A \rightarrow D$ is strictly singular if given any infinite dimensional subspace $A' \subset A$ and $\varepsilon > 0$ there is a further infinite dimensional subspace $A'' \subset A'$ such that $\|t|_{A''}\| < \varepsilon$. Both things together yield that given $B' \subset B$ there is $A'' \subset A' \subset A$ such that $I + K : A'' \rightarrow B'$ is an into isomorphism and $\|t|_{A''}\| < \varepsilon$. There is no loss of generality assuming that $\|K|_{A''}\| < \varepsilon$. Therefore $\|T|_{(I+K)(A'')}\| = \|t|_{A''} + TK|_{A''}\| < (1 + \|T\|)\varepsilon$. \square

We thus obtain the “strictly singular counterpart” to Corollary 2:

PROPOSITION 3. *If the natural quotient map $\mathcal{X}^{(2)} \rightarrow \mathcal{X}^{(1)}$ is strictly singular, then so is $\mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(k)}$ for every $n > k$.*

PROOF. Note that if $n > m > k$, then $\mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(k)}$ is $\mathcal{X}^{(n)} \rightarrow \mathcal{X}^{(m)}$ followed by $\mathcal{X}^{(m)} \rightarrow \mathcal{X}^{(k)}$. As the composition of a strictly singular operator with any operator is again strictly singular, we have that the Proposition is trivial if $k = 1$ and also that one can assume $n = k + 1$. We shall prove that $\mathcal{X}^{(k+1)} \rightarrow \mathcal{X}^{(k)}$ is strictly singular by induction on $k \in \mathbb{N}$. There is nothing to prove for $k = 1$, so assume $k > 1$. Since one has the commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{X}^{(1)} & \xlongequal{\quad} & \mathcal{X}^{(1)} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{X}^{(k)} & \longrightarrow & \mathcal{X}^{(k+1)} & \xrightarrow{\pi_{k+1,1}} & \mathcal{X}^{(1)} & \longrightarrow & 0 \\ & & \downarrow \pi_{k,k-1} & & \downarrow \pi_{k+1,k} & & \parallel & & \\ 0 & \longrightarrow & \mathcal{X}^{(k-1)} & \longrightarrow & \mathcal{X}^{(k)} & \longrightarrow & \mathcal{X}^{(1)} & \longrightarrow & 0. \end{array}$$

But $\pi_{k+1,1}$ is strictly singular and so is $\pi_{k,k-1}$, by the induction hypothesis. Thus, the result follows from Lemma 3. \square

We omit the proofs of the dual results:

LEMMA 4. Assume one has a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{I} & B & \xrightarrow{Q} & C & \longrightarrow & 0 \\ & & \parallel & & \uparrow T & & \uparrow t & & \\ 0 & \longrightarrow & A & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

with exact rows. If both I and t are strictly cosingular then T is strictly cosingular.

COROLLARY 3. If the inclusion map $\mathcal{X}^{(1)} \rightarrow \mathcal{X}^{(2)}$ is strictly cosingular, then so is $\mathcal{X}^{(k)} \rightarrow \mathcal{X}^{(n)}$ for every $k < n$.

5. Applications to Hilbert spaces

5.1. The quasi-linear map associated to twisted Kalton-Peck spaces. Some results in this section are, essentially, in [20, Section 6.B]. Let us consider the following variation of the Calderón space associated to the Banach couple (ℓ_∞, ℓ_1) which is designed to simplify the computation of extremals. Take $U = \mathbb{S} = \{z \in \mathbb{C} : 0 < \Re z < 1\}$, with $W = \ell_\infty$, and let \mathcal{F} be the space of analytic functions $F : \mathbb{S} \rightarrow \ell_\infty$ having the following properties:

- (1) F extends to a $\sigma(\ell_\infty, \ell_1)$ continuous function on $\bar{\mathbb{S}}$ that we denote again by F .
- (2) $\|F\|_{\mathcal{F}} = \sup\{\|F(it)\|_\infty, \|F(1+it)\|_1 : t \in \mathbb{R}\} < \infty$.

Let $(\mathcal{Z}_z)_{z \in \mathbb{S}}$ denote the analytic family induced by \mathcal{F} . Then of course $\mathcal{Z}_z = [\ell_\infty, \ell_1]_\theta = \ell_p$, where $\theta = \Re z$ and $p = 1/\theta$ for $\theta \in (0, 1)$ and, in particular $\mathcal{Z}_z = \ell_2$ for $z = 1/2$. In the remainder of this Section we fix $z = 1/2$ as the base point.

If x is normalized in ℓ_2 , then $F_x(z) = u|x|^{2z}$ is normalized in \mathcal{F} (although it does not belong to $\mathcal{C}(\ell_\infty, \ell_1)$ in general) and one has $F_x(\frac{1}{2}) = x$, where $x = u|x|$ is the “polar decomposition” of x . Now

$$F_x = u|x||x|^{2z-1} = x|x|^{2(z-1/2)} = x \sum_{n=0}^{\infty} \frac{2^n \log^n |x|}{n!} \left(z - \frac{1}{2}\right)^n,$$

and

$$\widehat{F}_x[n, \tfrac{1}{2}] = \frac{2^n x \log^n |x|}{n!},$$

if $\|x\|_2 = 1$. For arbitrary $x \in \ell_2$ we have, by homogeneity,

$$(10) \quad \widehat{F}_x[n, \tfrac{1}{2}] = \frac{2^n x}{n!} \log^n \left(\frac{|x|}{\|x\|_2} \right).$$

In particular,

$$\begin{aligned} \Omega_{1,n}(x) &= (F_x[n-1, \tfrac{1}{2}], \dots, F_x[1, \tfrac{1}{2}]) \\ &= x \left(\frac{2^{n-1}}{(n-1)!} \log^{n-1} \left(\frac{|x|}{\|x\|_2} \right), \dots, 2 \log^2 \left(\frac{|x|}{\|x\|_2} \right), 2 \log \left(\frac{|x|}{\|x\|_2} \right) \right) \end{aligned}$$

which allows us to describe the corresponding spaces $\mathcal{Z}^{(n)}$ for small n as follows. First, we have

$$\mathcal{Z}^{(2)} \approx \ell_2 \oplus_{\Omega_{1,1}} \ell_2 = \{(y, x) : \|y - 2x \log(|x|/\|x\|_2)\|_2 + \|x\|_2 < \infty\},$$

which is well isomorphic to Kalton-Peck Z_2 space [14, Section 6]. Also,

$$\mathcal{Z}^{(3)} \approx \mathcal{Z}^{(2)} \oplus_{\Omega_{1,2}} \ell_2 \approx (\ell_2 \oplus_{\Omega_{1,1}} \ell_2) \oplus_{\Omega_{1,2}} \ell_2,$$

and the norm of $\mathcal{Z}^{(3)}$ is equivalent to

$$(11) \quad \|(z, y, x)\|_{\Omega_{1,2}} = \left\| \left(z - 2x \log^2 \left(\frac{|x|}{\|x\|_2} \right), y - 2x \log \left(\frac{|x|}{\|x\|_2} \right) \right) \right\|_{\Omega_{1,1}} + \|x\|_2.$$

We will also finally display the quasilinear map $\Omega_{2,2}$ that allows one to represent $\mathcal{Z}^{(4)}$ as a twisted sum of $\mathcal{Z}^{(2)}$ with itself. After all, this was the starting point of this research. Let $\varphi : \mathbb{S} \rightarrow \mathbb{D}$ be conformal equivalence vanishing at $z_0 = 1/2$ and let $\phi = \sum_{1 \leq i \leq 3} a_i \varphi^i$ be such that $\hat{\phi}[i; z = \frac{1}{2}] = \delta_{1i}$ for $0 \leq i \leq 3$. Given $(y, x) \in \mathcal{Z}^{(2)}$ we construct an allowable $F_{(y,x)} \in \mathcal{F}$ as follows. Let F_x and $F_{(y-\Omega(x))}$ be extremals for x and $y - \Omega(x)$, respectively, where $\Omega(x) = \Omega_{1,1}(x) = F'_x(\frac{1}{2}) = 2x \log(|x|/\|x\|_2)$. Put

$$G = \phi \cdot F_{(y-\Omega(x))} + F_x.$$

Then $G(\frac{1}{2}) = x$, $G'(\frac{1}{2}) = y$ and $\|G\|_{\mathcal{F}} \leq \|\phi \cdot F_{(y-\Omega(x))}\|_{\mathcal{F}} + \|F_x\|_{\mathcal{F}} \leq \|\phi\|_{\infty}(\|y - \Omega(x)\|_2 + \|x\|_2)$, where $\|\phi\|_{\infty} \leq |a_1| + |a_2| + |a_3|$ and we may define

$$\Omega_{2,2}(y, x) = \left(\hat{F}_{(y,x)}[3; \tfrac{1}{2}], \hat{F}_{(y,x)}[2; \tfrac{1}{2}] \right).$$

By the construction of ϕ we have $\hat{F}_{(y,x)}[2; \frac{1}{2}] = \hat{F}_{(y-\Omega(x))}[1; \frac{1}{2}] + \hat{F}_x[2; \frac{1}{2}]$ and $\hat{F}_{(y,x)}[3; \frac{1}{2}] = \hat{F}_{(y-\Omega(x))}[2; \frac{1}{2}] + \hat{F}_x[3; \frac{1}{2}]$ and thus

$$\Omega_{2,2}(y, x) = 2 \left((y - \Omega x) \log^2 \frac{|y - \Omega x|}{\|y - \Omega x\|_2} + \frac{2x}{3} \log^3 \frac{|x|}{\|x\|_2}, (y - \Omega x) \log \frac{|y - \Omega x|}{\|y - \Omega x\|_2} + x \log^2 \frac{|x|}{\|x\|_2} \right).$$

5.2. The 3-space problem for twisted Hilbert spaces. We are now ready for the first concrete application. Recall that a twisted Hilbert space is a twisted sum of Hilbert spaces.

PROPOSITION 4. *The space $\mathcal{Z}^{(n)}$ is a twisted Hilbert space if and only if $n = 1, 2$.*

PROOF. The n -th cotype 2 constant $a_{n,2}(X)$ of a (quasi-) Banach space X is defined as the infimum of those C such that for every $x_1, \dots, x_n \in X$ one has

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \leq C \left(\sum_{i=1}^n \|x_i\|^2 \right)^{1/2},$$

where (r_n) is the sequence of Rademacher functions.

To prove that $\mathcal{Z}^{(3)}$ does not embed in any twisted Hilbert space we will work with the equivalent quasinorm given by (11). Let (e_i) be the unit basis of ℓ_2 and take $x_i = (0, 0, e_i)$. These are normalized vectors, which makes $(\sum_{i=1}^n \|x_i\|^2)^{1/2} = \sqrt{n}$. On the other hand

$$\left\| \sum_{i=1}^n \pm x_i \right\|_{\Omega_{1,2}} = \sqrt{n}(1 + \log^2 n).$$

Hence the cotype 2 constants of $\mathcal{Z}^{(3)}$ cannot verify $a_{n,2} \leq K \log n$. And this estimate must hold in any twisted Hilbert space by [14, Theorem 6.2.(a)]. \square

COROLLARY 4. *“To be a twisted Hilbert space” is not a 3-space property.*

The corollary answers a question posed to us by David Yost long time ago [5, p. 95] and considered by the first author in [1] where it was shown that “to be a subspace of a twisted Hilbert space” is not a 3-space property. Since $\mathcal{Z}^{(n)}$ is isomorphic to its dual (see [20, Section 4]) and the dual of any twisted Hilbert space is again a twisted Hilbert space, we see that $\mathcal{Z}^{(n)}$ is a quotient of a twisted Hilbert space if and only if $n = 1, 2$.

Thus, in the situation described in Section 5.1, recall that for $\mathcal{F} = \mathcal{F}(\ell_\infty, \ell_1)$ one gets $\mathcal{Z}^{(1)} = \ell_2$ and $\mathcal{Z}^{(2)}$ is isomorphic to Kalton-Peck’s space Z_2 and, actually, the extension $0 \rightarrow \ell_2 \rightarrow \mathcal{Z}^{(2)} \rightarrow \ell_2 \rightarrow 0$ is isomorphically (and even “projectively” cf. [14]) equivalent to Kalton-Peck’s sequence $0 \rightarrow \ell_2 \rightarrow Z_2 \rightarrow \ell_2 \rightarrow 0$, which has strictly singular quotient map and strictly cosingular inclusion (see [14, Theorem 6.4]). One therefore has.

PROPOSITION 5. *The exact sequences $0 \rightarrow \mathcal{Z}^{(k)} \rightarrow \mathcal{Z}^{(n+k)} \rightarrow \mathcal{Z}^{(n)} \rightarrow 0$ are singular and cosingular, for all integers n, k .* \square

As a direct application we get:

PROPOSITION 6. *If $k > 1$ the space $\mathcal{Z}^{(k)}$ does not contain complemented copies of ℓ_2 .*

PROOF. By [14, Corollary 6.7] $\mathcal{Z}^{(2)} = Z_2$ has no complemented subspaces isomorphic to ℓ_2 . Now, if one has an exact sequence

$$0 \longrightarrow A \xrightarrow{I} B \xrightarrow{Q} C \longrightarrow 0.$$

with Q strictly singular and A not containing ℓ_2 complemented then B does not contain ℓ_2 complemented: assume otherwise that B has a subspace B' which is isomorphic to ℓ_2 and is complemented in B through a projection P . (Without loss of generality we may assume that $A = \ker Q$ and I is the inclusion map.) Since Q is strictly singular, there exist an infinite dimensional subspace $A' \subset A$ and a nuclear operator $K : A' \rightarrow B$ such that $I - K : A' \rightarrow B'$ is an embedding. Passing to a further subspace if necessary we may assume the nuclear norm of K is strictly less than 1. Let N be a nuclear endomorphism of B extending K and having the same nuclear norm as K . Then $\|N : B \rightarrow B\| < 1$ and $\mathbf{1}_B - N$ is invertible, with $(\mathbf{1}_B - N)^{-1} = \sum_{k \geq 0} N^k$ – summation in the operator norm. Now, it is easily seen that

$$(\mathbf{1}_B - N) \circ P \circ (\mathbf{1}_B - N)^{-1}$$

is a projection of B (hence of A) onto A' . \square

The proof also works replacing ℓ_2 by any other “complementably minimal” space (those Banach spaces all whose infinite dimensional closed subspaces contain subspaces isomorphic to and complemented in the whole space) such as ℓ_p for $1 < p < \infty$. This implies that Proposition 6 extends almost verbatim for $1 < p < \infty$.

5.3. A twisted sum of Z_2 containing ℓ_2 complemented. It is quite surprising that there exists a twisted sum of Z_2 containing complemented copies of ℓ_2 . But they exist:

PROPOSITION 7. *There is a (nontrivial) exact sequence*

$$0 \longrightarrow Z_2 \longrightarrow \ell_2 \oplus \mathcal{X}^{(3)} \longrightarrow Z_2 \longrightarrow 0.$$

PROOF. Recall from [6, p.257] the construction of the so-called diagonal push-out sequence: In a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\iota} & B & \longrightarrow & E \longrightarrow 0 \\ & & \downarrow u & & \downarrow v & & \parallel \\ 0 & \longrightarrow & C & \xrightarrow{j} & D & \longrightarrow & E \longrightarrow 0 \end{array}$$

the following sequence is exact

$$0 \longrightarrow A \xrightarrow{\iota \times u} B \oplus C \xrightarrow{v \ominus j} D \longrightarrow 0,$$

where $(\iota \times u)(a) = (\iota(a), u(a))$ and $(v \ominus j)(b, c) = v(b) - j(c)$. Thus, taking $n = i = 1$ and $k = j = 2$ in Diagram 6 for $\mathcal{F} = \mathcal{F}(\ell_\infty, \ell_1)$ one gets a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Z}^{(2)} & \longrightarrow & \mathcal{Z}^{(3)} & \longrightarrow & \ell_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \ell_2 & \longrightarrow & \mathcal{Z}^{(2)} & \longrightarrow & \ell_2 \longrightarrow 0, \end{array}$$

from which, recalling that $\mathcal{X}^{(2)} = Z_2$, one obtains an exact sequence

$$0 \longrightarrow Z_2 \longrightarrow \ell_2 \oplus \mathcal{Z}^{(3)} \longrightarrow Z_2 \longrightarrow 0$$

which is not trivial since otherwise $Z_2 \simeq Z_2 \oplus Z_2 = \ell_2 \oplus \mathcal{Z}^{(3)}$, something impossible since Z_2 does not contain ℓ_2 complemented. \square

Therefore $\ell_2 \oplus \mathcal{Z}^{(3)}$ is a twisted sum of Z_2 , which contains complemented Hilbert subspaces. We cannot resist to remark that while nobody knows whether Z_2 is isomorphic to its hyperplanes, it is obvious that $\ell_2 \oplus \mathcal{Z}^{(3)}$ is isomorphic to its own hyperplanes.

6. Open ends

6.1. On the splitting of the first extension. Very little is known about the splitting of the “first” exact sequence $0 \rightarrow \mathcal{Z}^{(1)} \rightarrow \mathcal{Z}^{(2)} \rightarrow \mathcal{Z}^{(1)} \rightarrow 0$ outside of the case in which it is induced by a couple of Banach lattices. On the other hand, Corollary 2 shows that once the first exact sequence obtained in an interpolation schema is nontrivial, the same happens to all the rest. Is it true the reciprocal? That is, suppose that $\mathcal{Z}^{(2)}$ is a trivial self-extension of $\mathcal{Z}^{(1)}$. Does it follow that the extensions $0 \rightarrow \mathcal{Z}^{(k)} \rightarrow \mathcal{Z}^{(n+k)} \rightarrow \mathcal{Z}^{(n)} \rightarrow 0$ are trivial for all values of n and k ?

6.2. Other twisted Hilbert spaces. Suppose we have a Banach space X_0 with a normalized basis that we use to consider X_0 inside ℓ_∞ . Take $X_1 = \overline{X'_0}$ the complex conjugate of the closure X'_0 of the subspace spanned by the coordinate functionals in X_0^* . Then (X_0, X_1) is a Banach couple, $[X_0, X_1]_{1/2}$ is a Hilbert space (see [18, around Theorem 3.1]), and thus $\mathcal{Z}^{(2)}$ is a twisted Hilbert space. We believe that $\mathcal{Z}^{(2)}$ is a Hilbert space if and only if $X_0 = \ell_2$.

6.3. Other interpolation methods. Most of the work done here can be reproduced for real interpolation by either the K or J methods as it can be deduced from the results in this paper and those in [4]. It would be interesting to know to what extent the same occurs for other interpolation methods.

6.4. About the vanishing of Ext^2 . A problem at the horizon, for us, was whether the second derived functor Ext^2 vanishes on Hilbert spaces, which can be understood as a twisted reading of a question of Palamodov for Fréchet spaces ([16, Section 12, Problem 6]).

Given Banach spaces A and D , one considers the set of all possible four-term exact sequences

$$(12) \quad 0 \longrightarrow A \xrightarrow{I} B \xrightarrow{U} C \xrightarrow{Q} D \longrightarrow 0.$$

Under a certain equivalence relation, which is not necessary to define here, the set of such four-term exact sequences becomes a linear space denoted by $\text{Ext}^2(D, A)$, whose zero is (the class of all exact sequences equivalent to)

$$0 \longrightarrow A \xlongequal{\quad} A \xrightarrow{0} D \xlongequal{\quad} D \longrightarrow 0.$$

It is important to realize that if we are given a short exact sequence of the form

$$(13) \quad 0 \longrightarrow A \xrightarrow{I} B \xrightarrow{P} E \longrightarrow 0$$

and another sequence of the form

$$(14) \quad 0 \longrightarrow E \xrightarrow{J} C \xrightarrow{Q} D \longrightarrow 0$$

then we may form a four-term sequence

$$(15) \quad 0 \longrightarrow A \xrightarrow{I} B \xrightarrow{U} C \xrightarrow{Q} D \longrightarrow 0$$

just taking $U = J \circ P$. This resulting “long” sequence will be zero in $\text{Ext}^2(D, A)$ if and only if (13) and (14) fit inside a commutative diagram

$$(16) \quad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \uparrow & & \uparrow & \\ & & & D & \xlongequal{\quad} & D & \\ & & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & C \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & E \longrightarrow 0 \\ & & & & \uparrow & & \uparrow \\ & & & & 0 & & 0 \end{array}$$

whose rows and columns are exact.

The skeptical reader will wonder how is this related to the main subject of the paper. Let μ a σ -finite measure on a measure space S and let L_0 be the space of all (complex) measurable functions on S , where we identify two functions if they agree almost everywhere. If X is a Köthe space on μ , then a centralizer on X is a homogeneous mapping $\Omega : X \rightarrow L_0$ having the following property: there is a constant $C = C(\Omega)$ such that, for every $f \in X$ and every $a \in L_\infty$, the difference $\Omega(af) - a\Omega(f)$ belongs to X and $\|\Omega(af) - a\Omega(f)\|_X \leq C\|a\|_\infty\|f\|_X$. Every centralizer is quasilinear and so it induces a twisted sum $X \oplus_\Omega X = \{(y, x) : x, y - \Omega(x) \in X\}$ which is quasinormed by the functional $\|(y, x)\|_\Omega = \|y - \Omega(x)\|_X + \|x\|_X$. A widely ignored result by Kalton states that if X is super-reflexive then one can construct an admissible space of analytic functions \mathcal{F} on a disc centered at the origin such that:

- $X = X_0^{(1)}$ (evaluation at 0) up to equivalent norm;
- $\Omega \approx \Omega_{1,1}$, where $\Omega_{1,1}$ is the corresponding “derivation” (see Section 3.5).

This means that for every $x \in X$ the difference $\Omega(x) - \Omega_{1,1}(x)$ falls in X and one has the estimate $\|\Omega(x) - \Omega_{1,1}(x)\|_X \leq K\|x\|_X$ for some constant K and every $x \in X$. Actually one can construct \mathcal{F} by using no more than three Köthe spaces on the boundary of the disc [12, Theorem 7.9]; if Ω is “real” in the sense that it takes real functions into real functions, then two Köthe spaces on a strip suffice [12, Theorem 7.6]. In particular since $X \oplus_\Omega X = X \oplus_{\Omega_{1,1}} X = X_0^{(2)}$, up to equivalent (quasi-) norms, we see that the self-extension induced by Ω fits into the commutative diagram (the operators $\iota_{n,k}$ are those appearing in Proposition 1):

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & X_0^{(3)} & \longrightarrow & X \oplus_\Omega X \longrightarrow 0 \\ & & \parallel & & \uparrow \iota_{2,3} & & \uparrow \iota_{1,2} \\ 0 & \longrightarrow & X & \longrightarrow & X \oplus_\Omega X & \longrightarrow & X \longrightarrow 0 \end{array}$$

which, when completed, has the same form as (16) witnessing that the juxtaposition of two copies of the extension induced by Ω , namely

$$0 \longrightarrow X \longrightarrow X \oplus_{\Omega} X \longrightarrow X \oplus_{\Omega} X \longrightarrow X \longrightarrow 0,$$

is zero in $\text{Ext}^2(X, X)$. We do not know what happens with two different centralizers; more specifically, we ask the following. Let Ω and Φ be centralizers on a super-reflexive Köthe space X and consider the twisted sums $X \oplus_{\Omega} X$ and $X \oplus_{\Phi} X$. If, as before, we set $I(x) = (x, 0)$, $U(x, y) = (y, 0)$ and $Q(x, y) = y$, can the exact sequence

$$0 \longrightarrow X \xrightarrow{I} X \oplus_{\Omega} X \xrightarrow{U} X \oplus_{\Phi} X \xrightarrow{Q} X \longrightarrow 0$$

be nonzero in $\text{Ext}^2(X, X)$?

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE EXTREMADURA, AVENIDA DE ELVAS, 06071-BADAJOS, SPAIN

E-mail address: `fcabello@unex.es`, `castillo@unex.es`